Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 308 (2008) 220–230
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

Introduction of measures for segments and angles in a general absolute plane

Helmut Karzel[a], Mario Marchi[b, 1]

[a] Zentrum Mathematik, T.U. München, D-80290 München, Germany
[b] Dipartimento di Matematica e Fisica, Università Cattolica, Via Trieste, 17, I-25121 Brescia, Italy

Received 31 October 2004; received in revised form 30 September 2005; accepted 27 November 2006
Available online 2 June 2007

Abstract

To an absolute plane $(E, \mathcal{L}, \equiv, \alpha)$ in the general sense of Karzel et al. [Einführung in die Geometrie, UTB 184, Vandenhoeck, Göttingen, 1973, Section 16] there will be associated an ordered commutative group $(W, +, <)$ such that $(W, +)$ is a subgroup of the corresponding K-loop $(E, +)$ of the absolute plane and a cyclic ordered commutative group $(E_1, \cdot, \zeta)$ where $(E_1, \cdot)$ is isomorphic to a rotation group fixing a point. $(W, +, <)$, resp. $(E_1, \cdot, \zeta)$, will serve to introduce a distance λ describing the congruence and satisfying the triangular inequality or resp. a measure μ for angles describing the congruence and conjugacy of angles.

Keywords: Absolute plane; Distance function; Measure of angles; Group; Loop

0. Introduction

Let (o, e_1, e_2) be a *frame of reference* of an absolute plane $(E, \mathcal{L}, \equiv, \alpha)$, let be $W := \overline{o, e_1}$ (the line joining the points o and e_1), $W_+ := \overrightarrow{o, e_1}$ (the halfline) and $E_1 := \{x \in E | (x, o) \equiv (e_1, o)\}$ the *unit circle* with center o and passing through e_1. For $a, b \in E$ with $a \neq b$ let $\widetilde{a, b}$, resp. $\widehat{a, b}$, be the reflection in the midpoint, resp. in the midline, of the points a and b and $\sigma_{a,b}$ the proper motion uniquely determined by $\sigma_{a,b}(a) = o$ and $\sigma_{a,b}(b) \in W_+$. If $a \in E$, $L \in \mathcal{L}$ let $\tilde{a}$, resp. $\tilde{L}$, be the reflection in the point a, resp. in the line L. Then $(E, +)$ with $a + b := \widetilde{o, a} \circ \tilde{o}(b)$ is a K-loop, W a commutative subgroup of $(E, +)$ and W^+ a positive domain of $(W, +)$ hence $(W, +, <)$ with "$a < b :\Longleftrightarrow b - a \in W_+$" an ordered group. Moreover there is an *absolute value*

$$| \ | : E \to W_+ \cup \{o\}; \ \ x \mapsto |x| := \sigma_{o,x}(x) \quad \text{if } x \neq o \text{ and } |o| := o$$

such that

$$\lambda : E \times E \to W_+ \cup \{o\}; \ \ (x, y) \mapsto \lambda(x, y) := | - x + y|$$

is a *distance* describing the congruence and satisfying the triangular inequality such that "$\lambda(a, b) = \lambda(a, c) + \lambda(c, b) \Longleftrightarrow c \in [a, b]$" ($[a, b]$ denotes the segment between a and b).

[1] Research partially supported by the Research Project of M.I.U.R. (Italian Ministry of Education, University and Research) "Strutture geometriche, combinatoria e loro applicazioni" and by the Research group G.N.S.A.G.A. of INDAM.

E-mail address: m.marchi@dmf.unicatt.it (M. Marchi).

0012-365X/$ - see front matter
doi:10.1016/j.disc.2006.11.036

For each $a \in E_1$ let $a^{\cdot} := \widehat{e_1, a} \circ \widetilde{W}$ if $a \neq e_1$ and let $e_1^{\cdot} := \mathrm{id}$. Then $(E_1, \cdot)$ with $a \cdot b := a^{\cdot}(b)$ is a commutative group isomorphic to each rotation group fixing a point. By setting $\zeta(x, y, z) := (\overline{e_1, x^{-1}z}|e_1 + e_2, x^{-1}y)$ for[2] $(x, y, z) \in \left(\binom{E_1}{3}\right)$ (cf. (1.3) and (5.3)) $(E_1, \cdot, \zeta)$ becomes a *cyclic ordered group*. Let $\alpha := \measuredangle(a, b, c)$ and $\beta := \measuredangle(d, e, f)$ be angles then the *oriented measure* μ of α will be an element of the cyclic ordered group $(E_1, \cdot, \zeta)$ given by $\{\mu(\alpha)\} := \overrightarrow{(o, \sigma_{b,a}(c))} \cap E_1$. If $\overrightarrow{b, c} = \overrightarrow{e, d}$ then α and β can be added, $\alpha + \beta := \measuredangle(a, b, f)$ is the *sum* and for the oriented measures we have $\mu(\alpha + \beta) = \mu(\alpha) \cdot \mu(\beta)$. Moreover α and β are *congruent* if and only if $\mu(\beta) \in \{\mu(\alpha), (\mu(\alpha))^{-1}\}$.

Finally, we remark that $E_1^{\cdot} := \{x^{\cdot}|x \in E_1\}$ is a commutative subgroup of the group $\mathscr{M}^+$ of all proper motions, that for each $a \in E$, $a^+ := \widetilde{o, a} \circ \tilde{o}$ is a proper motion and that for each $\tau \in \mathscr{M}^+$ there is exactly one pair $(a, s) \in E \times E_1$ such that $\tau = a^+ \circ s^{\cdot}$. Moreover if $(a, s), (b, t) \in E \times E_1$ then $(a^+ \circ s^{\cdot}) \circ (b^+ \circ t^{\cdot}) = (a + s^{\cdot}(b))^+ \circ (\delta_{a,s^{\cdot}(b)} \circ s^{\cdot} \circ t^{\cdot})$ with $\delta_{a,b} := ((a + b)^+)^{-1} \circ a^+ \circ b^+ \in E_1^{\cdot}$. That means that $\mathscr{M}^+ = (E, +) \times_Q E_1^{\cdot}$ is the *quasidirect product* of the K-loop $(E, +)$ and the commutative group $(E_1, \cdot)$ (cf. e.g. [7]).

1. Axiomatic basis

In this paper $(E, \mathscr{L}, \equiv, \alpha)$ will be an *absolute plane* in the sense of [6, Section 16] where E, resp. $\mathscr{L}$, denotes the set of *points*, resp. *lines*, $\equiv$ the *congruence relation* between pairs of points and α the *order structure*. We make no claim concerning continuity or the Archimedian axiom. Our axiomatic assumptions can also be described as follows (cf. e.g. [4]):

(1.1) $(E, \mathscr{L})$ is an *incidence space* (or *linear space*). If $a, b \in E$ with $a \neq b$ then we denote by $\overline{a, b}$ the uniquely determined line joining a and b. A subset $T \subseteq E$ is called *subspace* of $(E, \mathscr{L})$ if $\forall x, y \in T, x \neq y : \overline{x, y} \subseteq T$. Let $\mathscr{T}$ be the set of all subspaces of $(E, \mathscr{L})$ and let $E^{(3)} := \{(x, y, z) \in E^3 | x \neq y, z \wedge z \in \overline{x, y}\}$.

(1.2) $(E, \mathscr{L}, \alpha)$ is an *ordered space*, i.e.

$$\alpha : E^{(3)} \to \{-1, 1\}, \quad (x, y, z) \mapsto (x|y, z)$$

is a map satisfying the following conditions:

(Z1) $\forall (a, b, c), (a, b, d) \in E^{(3)} : (a|b, c) \cdot (a|c, d) = (a|b, d)$.
(Z2) For $a, b, c \in E$ distinct and collinear (i.e. $c \in \overline{a, b}$) exactly one of the values $(a|b, c)$, $(b|c, a)$, $(c|a, b)$ equals -1.
(Z3) $\forall a, b \in E, a \neq b \ \exists c \in E$ such that $b \in]a, c[:= \{x \in E \mid (x, a, c) \in E^{(3)} \wedge (x|a, c) = -1\}$.
(ZP) If $a, b, c \in E$ are noncollinear, $x \in]b, c[$ and $y \in]a, x[$ then $\overline{c, y} \cap]a, b[\neq \emptyset$.[3]

If $a, b \in E, a \neq b$ and $T \in \mathscr{T}$ with $a \notin T$ we can introduce besides the open segment $]a, b[$ notions as "convex subset", *halfspace* $\overrightarrow{T, a}^{-} := \{x \in E \backslash T | \ ]x, a[\cap T \neq \emptyset\}$ and in particular *halfline* $\overrightarrow{b, a}^{-}$. Also $\overrightarrow{b, a} := \{x \in \overline{a, b} | (b|a, x) = 1\}$ is a halfline. Let $\mathscr{H}$ be the set of all halflines. By [2, Theorem 1.5] $(E, \mathscr{L})$ is an exchange space and therefore we can claim:

(1.3) $(E, \mathscr{L}, \boldsymbol{\alpha})$ is an *ordered plane*, i.e. $\dim(E, \mathscr{L}) = 2$. Here we can define the following *order function* in the sense of Sperner [9]. If $(\mathscr{L} \times E \times E)' := \{(L, a, b) \in \mathscr{L} \times E \times E | a, b \notin L\}$ let:
$(\ | \ , \ ) : (\mathscr{L} \times E \times E)' \to \{1, -1\}$; $(L, a, b) \mapsto (L|a, b)$ with $(L|a, b) = -1$ if $L \cap]a, b[\neq \emptyset$ and $(L|a, b) = +1$ otherwise.

Hence $(\ | \ , \ )$ is a function which maps $(\mathscr{L} \times E \times E)'$ in the cyclic group $(\{1, -1\}, \cdot)$ of order 2. This function has the properties:

(1.4) $\forall L \in \mathscr{L}, \forall a, b, c \in E \backslash L$:

(O1) $(L|a, b) \cdot (L|b, c) = (L|a, c)$.

---

[2] If E is a set, $\binom{E}{3} := \{\{x, y, z\} \subseteq E \mid |\{x, y, z\}| = 3\}$ denotes the set consisting of all 3-sets of E and $\left(\binom{E}{3}\right) := \{(x, y, z) \in E^3 \mid \{x, y, z\} \in \binom{E}{3}\}$ denotes the set of all the triples (x, y, z) of E such that $\{x, y, z\}$ is a 3-set.
[3] This statement is equivalent to the usual formulation of the axiom of Pasch.

(O2) If $a, b, c \in E$ are collinear if $A, B, C \in \mathscr{L}$ such that $a \in A,\ b, c \notin A,\ b \in B,\ a, c \notin B$ and $c \in C,\ a, b \notin C$ then exactly one of the values $(A|b, c), (B|c, a), (C|a, b)$ is equal -1.
(O3) $]a, b[\ \cap L \neq \emptyset \Longleftrightarrow (L|a, b) = -1$.
(O4) If a, b, c are collinear, if $G, H \in \mathscr{L}$ with $c \in G, H$ and $a, b \notin G \cup H$, then $(G|a, b) = (H|a, b)$.

(1.5) $(E, \mathscr{L}, \equiv)$ is a *plane with congruence* in the sense of Sörensen [8] hence: the congruence "$\equiv$" is an equivalence relation such that for all $a, b, c \in E : (a, b) \equiv (b, a)$ and "$(a, a) \equiv (b, c) \Longleftrightarrow b = c$" and such that the following compatible axioms (W1), (W2) and (W3) are valid:

(W1) $\forall a, b, c \in E$ collinear and distinct, $\forall a', b' \in E$ with $(a, b) \equiv (a', b')\ \exists_1\ c' \in \overline{a', b'} : (a, b, c) \equiv (a', b', c')$ (i.e. $(a, b) \equiv (a', b') \wedge (b, c) \equiv (b', c') \wedge (a, c) \equiv (a', c')$).

(W2) $\forall a, b, x \in E$ noncollinear, $\forall a', b', x' \in E$ with $(a, b, x) \equiv (a', b', x')$, $\forall c \in \overline{a, b}$, $\forall c' \in \overline{a', b'}$ with $(a, b, c) \equiv (a', b', c')$ we have $(x, c) \equiv (x', c')$.

(W3) $\forall a, b, x \in E$ noncollinear there exists exactly one $x' \in E$ with $x' \neq x$ and $(a, b, x) \equiv (a, b, x')$, denoted by $\widetilde{\overline{a, b}}(x) := x'$.

(1.6) $(E, \mathscr{L}, \equiv)$ is a plane with congruence satisfying the further axiom:

(V2) $\forall a, b \in E$ with $a \neq b$, $\forall G \in \mathscr{L}$, $\forall c \in G$ there are exactly two distinct points $d, d' \in G$ with $(c, d) \equiv (c, d') \equiv (a, b)$.

Thus $(E, \mathscr{L}, \equiv, \alpha)$ is an *absolute plane* if $(E, \mathscr{L}, \alpha)$ is an ordered plane, if $(E, \mathscr{L}, \equiv)$ is a plane with congruence satisfying (V2) and if it is fulfilled the compatibility axiom:

(V3) If $a, b, c \in E$ are distinct and collinear with $(a, b) \equiv (a, c)$ then $(a|b, c) = -1$.

By [6], (13.12), we have:

(1.7) Let $(E, \mathscr{L}, \equiv, \alpha)$ be an absolute plane, let $A, B, C \in \mathscr{L}$ be three distinct lines incident with a common point p and let $a \in A\backslash\{p\}, b \in B\backslash\{p\}, c \in C\backslash\{p\}$ such that $(A|b, c) = 1$, then $(B|a, c) = -(C|a, b)$.

2. Properties of absolute planes

Now we collect properties of the absolute plane $(E, \mathscr{L}, \equiv, \alpha)$ which will be used in order to introduce a measure for segments and a measure for angles.

For $a \in E$, resp. $L \in \mathscr{L}$, let $\tilde{a}$, resp. $\widetilde{L}$, denote the reflection in the point a, resp. in the line L, and let $\widetilde{E} := \{\tilde{p} | p \in E\}$, $\widetilde{\mathscr{L}} := \{\widetilde{L} | L \in \mathscr{L}\}$. Let $\mathscr{M}$ denote the set of all motions of $(E, \mathscr{L}, \equiv, \alpha)$, then:

(2.1) $\mathscr{M} = \widetilde{\mathscr{L}} \circ \widetilde{\mathscr{L}} \dot{\cup} \widetilde{\mathscr{L}} \circ \widetilde{\mathscr{L}} \circ \widetilde{\mathscr{L}}$ and so $\mathscr{M}^+ := \widetilde{\mathscr{L}} \circ \widetilde{\mathscr{L}}$ is a subgroup of index 2 in the group $(\mathscr{M}, \circ)$. The elements of $\mathscr{M}^+$ are called *proper motions*.

By [6], (16.11), resp. (16.12), any two distinct points $a, b \in E$ have exactly one *midpoint* c, resp. *midline* C, i.e. $\tilde{c}(a) = b$, resp. $\widetilde{C}(a) = b$, and we have $c \in C,\ C \perp \overline{a, b},\ \tilde{c} = \widetilde{C} \circ \widetilde{\overline{a, b}}$. We set $\widehat{a, b} := \widetilde{C},\ \widetilde{a, b} := \tilde{c}$ and $\tilde{a} := \widetilde{a, a}$.

By [6], (16.9), $\forall p \in E, \forall G \in \mathscr{L}, \exists_1 H \in \mathscr{L}$ with $p \in H$ and $H \perp G$; we set $(p \perp G) := H$. Therefore:

(2.2) $\forall A, B, C \in \mathscr{L}\ \forall x, y \in B$ with $A \neq B$ and $A, B \perp C$:

(1) $A \cap B = \emptyset$.
(2) $(A|x, y) = 1$.

We remark that (2.2.2) follows from (2.2.1) and (1.4, O3).

By [6], (17.2):

(2.3) $\forall a, b \in E, \forall L \in \mathscr{L}$ with $a \neq b$ and $b \notin L$:

(1) $(L|b, \widetilde{L}(b)) = -1$.
(2) $(a|b, \tilde{a}(b)) = -1$.
(3) $\widetilde{L}$ is order preserving.

From [6], (17.15) we obtain:

(2.4) $(\mathscr{H}, \mathscr{M}^+)$ is a regular permutation group, i.e. to any two halflines H_1 and H_2 there is exactly one proper motion mapping the halfline H_1 onto the halfline H_2.

3. The measure groups

In order to measure the "length" of segments resp. the "size" of angles we define an ordered commutative group $(W, +, <)$ resp. a cyclic ordered commutative group $(E_1, \cdot, \zeta)$.

Let (o, e_1, e_2) be a *frame of reference*, i.e. three noncollinear points of E such that $(o, e_1) \equiv (o, e_2)$ and $\overline{o, e_1} \perp \overline{o, e_2}$. Moreover let $W := \overline{o, e_1}$, $W_+ := \overrightarrow{o, e_1} = \{x \in W | (o|e_1, x) = 1\}$ and $E_1 := \{x \in E | (o, x) \equiv (o, e_1)\}$.

For each $a \in E$ let $a^+ := \widetilde{o, a} \circ \tilde{o}$ and for each $c \in E_1$ let $c^{\cdot} := \widehat{e_1, c} \circ \widetilde{W}$ if $c \neq e_1$ and $e_1^{\cdot} := \mathrm{id}$ (if $c = e_1$). Then $a^+, c^{\cdot} \in \mathscr{M}^+$, a^+ is fixedpointfree if $a \neq o$ and $c^{\cdot}$ is a rotation in o. Then $E_+ := \{p^+ | p \in E\}$ consists of the identity $o^+ = \mathrm{id}$ and fixedpointfree proper motions and $E_1^{\cdot} := \{c^{\cdot} | c \in E_1\}$ is the group of all rotations in o. By setting

$$+ : E \times E \to E;\ (a, b) \mapsto a + b := a^+(b) = \widetilde{o, a} \circ \tilde{o}(b) \quad \text{and}$$

$$\cdot : E_1 \times E_1 \to E_1;\ (a, b) \mapsto a \cdot b := a^{\cdot}(b) = \widehat{e_1, a} \circ \widetilde{W}(b)$$

$(E, +)$ becomes a K-loop (cf. e.g. [2]) and $(E_1, \cdot)$ a commutative group isomorphic to the rotation group in o. Moreover each line L containing o, in particular W, is a commutative subgroup of the loop $(E, +)$ and for each $x \in E$ we have $x + \tilde{o}(x) = o$. Therefore $-x := \tilde{o}(x)$ is the negative element of x.

For each $x \in E_1$ we have $x \cdot e_1 = \widehat{e_1, x} \circ \widetilde{W}(e_1) = \widehat{e_1, x}(e_1) = x$, $e_1 \cdot x = e_1^{\cdot}(x) = \mathrm{id}(x) = x$ (i.e. e_1 is the neutral element of $(E_1, \cdot)$) and $x \cdot \widetilde{W}(x) = \widehat{e_1, x} \circ \widetilde{W}(\widetilde{W}(x)) = \widehat{e_1, x}(x) = e_1$ thus $x^{-1} := \widetilde{W}(x)$ is the inverse element of x in $(E_1, \cdot)$.

By [2] we have:

(3.1) W_+ is a *positive domain* of the group $(W, +)$, i.e. $W = -W_+ \dot{\cup} \{o\} \dot{\cup} W_+$ and $W_+ + W_+ \subseteq W_+$ and so if we set for $a, b \in W$:

$$a < b :\Longleftrightarrow -a + b \in W_+$$

then $(W, +, <)$ is an ordered group.

Proof. For each $x \in E^* := E \backslash \{o\}$ we have by (2.3.2) $(o|x, \tilde{o}(x)) = -1$ and $-x = \tilde{o}(x)$ hence $(o|x, -x) = -1$. Therefore $W_- := \tilde{o}(W_+) = -(W_+)$ showing $W = -(W_+) \dot{\cup} \{o\} \dot{\cup} W_+$. Finally, let $a, b \in W_+$ then by (Z1) $(o|a, b) = (o|e_1, a) \cdot (o|e_1, b) = 1 \cdot 1 = 1$ and so by applying the motion $\tilde{o}$ $(o|\tilde{o}(a), \tilde{o}(b)) = 1$. On the other hand, $(o|a, \tilde{o}(a)) = -1$ (by (2.3.2)) thus $(o|a, \tilde{o}(b)) = (o|a, \tilde{o}(a)) \cdot (o|\tilde{o}(a), \tilde{o}(b)) = (-1) \cdot 1 = -1$ and by applying the motion $\widetilde{o, a}$, $-1 = (a|o, \widetilde{o, a}(\tilde{o}(b))) = (a|o, a + b)$. Hence by (Z2) and (Z1), $1 = (o|a, a + b) = (o|e_1, a) \cdot (o|e_1, a + b) = 1 \cdot (o|e_1, a + b)$, i.e. $a + b \in W_+$. $\square$

(3.2) Let $u, v \in W$ and $o < u$, then:

$$v \in]o, u[\Longleftrightarrow o < v < u. \tag{1}$$

Proof. "$\Rightarrow$": $v \in]o, u[\Longleftrightarrow (v|o, u) = -1 \Rightarrow$ (by (Z2))$(o|v, u) = 1 \Longleftrightarrow v \in W_+ \Longleftrightarrow o < v$. By (2.3.2), $-1 = (o|v, \tilde{o}(v)) = (o|v, -v)$ and since $(-v)^+$ is a motion hence betweenness preserving, $-1 = (v|o, u) = (o|-v, -v+u)$. Together we obtain $(o|v, -v+u) = (o|v, -v) \cdot (o|-v, -v+u) = (-1) \cdot (-1) = 1$ thus $-v + u \in W_+ \Longleftrightarrow v < u$.

"$\Leftarrow$": Now $o < v < u$ implies $u, v \in W_+$ hence $(o|v, u) = 1$ and $o < -v + u$, i.e. by (2.3.2) $1 = (o|u, -v+u) = (o|-u, -u+v) = (u|o, v)$. Now $1 = (o|v, u) = (u|o, v)$ implies by (Z2), $(v|o, u) = -1$, i.e. $v \in]o, u[$. $\square$

(3.3) Let $e_o := e_1 + e_2$ and $\overrightarrow{W, e_2} := \{x \in E \backslash W \mid]x, e_2[\cap W \neq \emptyset\}$, let $x \in E_1 \backslash \{e_1\}$, $X := (o \perp \overline{e_1, x})$ and $y \in E_1 \backslash \{e_1, x\}$, then:

(1) $\widehat{e_1, x} = \widetilde{X}$ so $x^{\cdot} = \widetilde{X} \circ \widetilde{W}$ and $x^{-1} = \widetilde{W}(x)$.
(2) $(\overline{e_1, x}|y, \widetilde{X}(y)) = 1$ hence $(\overline{e_1, x}|y, xy^{-1}) = 1$.
(3) $e_o \in \overrightarrow{W, e_2} \cap (e_1 \perp W)$.

Proof. (1) Since $\overline{(o, e_1)} \equiv (o, x)$, by [6] (16.13), X is the midline of e_1 and x and so $\widehat{e_1, x} = \widetilde{X}$.

(2) By $\overline{e_1, x}, \overline{y, \widetilde{X}(y)} \perp X$ and $y \neq e_1, x$ follows $\overline{e_1, x} \cap \overline{y, \widetilde{X}(y)} = \emptyset$ (cf. [6] (16.10.2)) and so by (2.2.2) $(\overline{e_1, x}|y, \widetilde{X}(y)) = 1$. Since by (1) $\widetilde{X}(y) = x^{\cdot} \circ \widetilde{W}(y) = x \cdot y^{-1}$ we obtain the assertion.

(3) Since $e_1^+ = \widetilde{o, e_1} \circ \tilde{o}$, e_1^+ is a proper motion fixing W and so also $\overrightarrow{W, e_2}$ and mapping $\overline{o, e_2} = (o \perp W)$ onto $(e_1 \perp W)$ hence $e_o = e_1 + e_2 \in (e_1 \perp W)$. □

4. Absolute value and distance

By (2.4) to any two distinct points $a, b \in E$ there is exactly one proper motion $\sigma_{a,b} \in \mathscr{M}^+$ with $\sigma_{a,b}(a) = o$ and $\sigma_{a,b}(b) \in W_+$. Let $\overline{W_+} := W_+ \cup \{o\}$. Therefore we can define:

the function

$$| \ | : E \to \overline{W_+};\ x \mapsto |x| := \sigma_{o,x}(x) \quad \text{if } x \neq o \text{ and } |o| := o$$

is called *absolute value* and

$$\lambda : E \times E \to \overline{W_+};\ (a, b) \mapsto \lambda(a, b) := |-a + b|$$

the *distance*.

(4.1) Let $a, b, c, d \in E$ and $\beta \in \mathscr{M}$ a motion, then:

(1) $\lambda(\beta(a), \beta(b)) = \lambda(a, b) = \lambda(b, a) = \sigma_{a,b}(b) \in \overline{W_+}$.
(2) $(a, b) \equiv (c, d) \iff \lambda(a, b) = \lambda(c, d)$ and "$\lambda(a, b) = o \iff a = b$".
(3) If $c \in]a, b[$ then $\lambda(a, b) = \lambda(a, c) + \lambda(c, b)$.

Proof. (1) If β is an improper motion then $\beta' := \beta \circ \widetilde{\overline{a, b}}$ is proper and $\beta(a) = \beta'(a)$, $\beta(b) = \beta'(b)$, therefore we may assume that β is proper, and so $\sigma_{\beta(a),\beta(b)} \circ \beta$ is a proper motion mapping a onto o and b onto a point $b' \in W_+$. Hence $\sigma_{a,b} = \sigma_{\beta(a),\beta(b)} \circ \beta$. In particular, since $(a^+)^{-1}$ is proper, $\sigma_{a,b} = \sigma_{o,-a+b} \circ (a^+)^{-1}$ and so $\lambda(a, b) = |-a + b| = \sigma_{o,-a+b}(-a + b) = \sigma_{a,b} \circ a^+(-a + b) = \sigma_{a,b}(b)$ and $\lambda(\beta(a), \beta(b)) = \sigma_{\beta(a),\beta(b)}(\beta(b)) = \sigma_{a,b}(b)$. Now if m is the midpoint of a and b, then $\widetilde{m}$ is proper and $\widetilde{m}(a) = b$, $\widetilde{m}(b) = a$ hence $\lambda(a, b) = \lambda(b, a)$.

(2) By [6], (17.15) there is a proper motion $\gamma \in \mathscr{M}^+$ with $\{\gamma(a), \gamma(b)\} = \{c, d\}$ if and only if $(a, b) \equiv (c, d)$.

(3) $c \in]a, b[$ implies $\overrightarrow{a, b} = \overrightarrow{a, c}$ hence $\sigma_{a,b} = \sigma_{a,c}$. Let $u := \sigma_{a,b}(b)$, $v := \sigma_{a,b}(c)$, $w := \sigma_{c,b}(b)$ then (since $\sigma_{a,b}$ is a motion with $\sigma_{a,b}(a) = o$), $v \in]o, u[$ hence by (3.2), $o < v < u$ and so by (1), $u = \lambda(o, u) = \lambda(a, b)$, $v = \lambda(o, v) = \lambda(a, c)$ and $w = \lambda(c, b) = \lambda(v, u) = |-v + u| = u - v \in W_+$. Consequently $\lambda(a, c) + \lambda(c, b) = v + (u - v) = u = \lambda(a, b)$. □

Let Δ be the set of all *triangles*, i.e. the set of all triples (a, b, c) of noncollinear points.

(4.2) Let $(a, b, c) \in \Delta$ be rectangular with $A := \overline{b, c} \perp \overline{a, c} =: B$, $C := \overline{a, b}$, let $u \in]a, b[$, $v \in]a, c[$, $\{u'\} := (u \perp B) \cap B$ and $\{v'\} := (v \perp C) \cap C$. Then:

(1) $\lambda(a, c), \lambda(b, c) < \lambda(a, b)$.
(2) $\lambda(u, u') < \lambda(b, c)$.
(3) $v' \in]a, b[$, $\lambda(v, v') < \lambda(b, c)$ and $\lambda(v, c) < \lambda(v', b)$.
(4) $\lambda(v, b) < \lambda(a, b)$.

Proof. (1) Let $d \in \overrightarrow{a, b}$ with $(a, d) \equiv (a, c)$ and let A' be the midline of c and d. Then $a \in A'$ (cf. [6], (16.13)), $\widetilde{A'}(c) = d$, $A' \perp \overline{c, d}$, $d \neq b$, $(A'|c, d) = -1$ (cf. [6], (16.11) and (16.13)) and $(A'|b, d) = (a|b, d) = 1$. Hence $(A'|b, c) = (A'|b, d) \cdot (A'|c, d) = 1 \cdot (-1) = -1$. This implies $\{a'\} := A' \cap]b, c[\neq \emptyset$ and $\{a''\} := A' \cap]d, c[\neq \emptyset$ hence $(a'|b, c) = (a''|d, c) = -1$. Now $(d|a, b) = (\overline{c, d}|a, b) = (\overline{c, d}|a, a') \cdot (\overline{c, d}|a', b)$. Since $\overline{c, d} = \overline{c, a''} \perp A' = \overline{a, a'}$ we have $(\overline{c, d}|a, a') = (a''|a, a') = -1$ by [6], (20.5.1) and since $(a'|b, c) = -1$, $(\overline{c, d}|a', b) = (c|a', b) = 1$ by (Z2). Thus $(d|a, b) = (-1) \cdot 1 = -1$, i.e. $d \in]a, b[$ implying $\lambda(a, c) = \lambda(a, d) < \lambda(a, b)$ by (4.1.3).

(2) Let M be the midline of c and $u', b' := \widetilde{M}(b)$, $D := \overline{b, b'}$ and $G := \overline{b, u'}$. Since $B, D \perp M$, $(B|b, b') = (D|a, u') = 1$. Since $u \in]a, b[$, i.e. $(u|a, b) = -1$ we obtain by (Z2): $1 = (a|b, u) = (B|b, u)$ and $1 = (b|a, u) = (D|a, u) = (G|a, u)$. Hence $(G|b', u) = (B|b', u) = (B|b', b) \cdot (B|b, u) = 1 \cdot 1 = 1$ and $(D|u, u') = (D|u, a) \cdot (D|a, u') = 1 \cdot 1 = 1$. Now $b \in C, D, G$ and $u \in C$, $b' \in D$, $u' \in G$ imply (by [6], (13.12.2)): $-1 = (C|b', u') \cdot (D|u, u') \cdot (G|u, b') = (C|b', u') \cdot 1 \cdot 1 = (C|b', u') = (u|b', u')$ hence $u \in]b', u'[$ and so (since $(u', b') \equiv (c, b)$) $\lambda(u, u') < \lambda(b', u') = \lambda(c, b)$.

(3) Let $V := (v \perp B)$ then (since $A \perp B$) $(V|b, c) = 1$ and (since $v \in]a, c[$) $(V|a, c) = (v|a, c) = -1$ hence $(V|a, b) = (V|b, c) \cdot (V|a, c) = 1 \cdot (-1) = -1$ and so $\{f\} := V \cap]a, b[$ exists. By (1) $\lambda(v, v') < \lambda(v, f)$, by (2) $\lambda(f, v) < \lambda(b, c)$ thus $\lambda(v, v') < \lambda(b, c)$. By [6], (20.5.1), $v' \in]a, f[\subset]a, b[$. Since $v \in]a, c[$, resp. $v' \in]a, b[$, we have by (4.1.3) $\lambda(a, v) + \lambda(v, c) = \lambda(a, c)$ or resp. $\lambda(a, v') + \lambda(v', b) = \lambda(a, b)$.

By (1), $\lambda(a, c) < \lambda(a, b)$ and $\lambda(a, v') < \lambda(a, v)$ thus $\lambda(v, c) = \lambda(a, c) - \lambda(a, v) < \lambda(a, b) - \lambda(a, v') = \lambda(v', b)$.

(4) Let $\{a'''\} := (a \perp \overline{b, v}) \cap \overline{b, v}$, $\{c'\} := (c \perp \overline{b, v}) \cap \overline{b, v}$ and $F := (v \perp \overline{b, v})$. Then (by [6], (20.5.1)) $(c'|b, v) = -1$ hence by (Z2) $(F|b, c') = (v|b, c') = 1$ and $(F|a, a''') = (F|c, c') = 1$. This implies: $(v|a''', b) = (F|a''', b) = (F|a''', b) \cdot (F|a, a''') \cdot (F|c', c) \cdot (F|b, c') = (F|a, c) = (v|a, c) = -1$ (since $v \in]a, c[$) and so $v \in]a''', b[$ thus $\lambda(v, b) < \lambda(a''', b)$ and by (1), $\lambda(a''', b) < \lambda(a, b)$. □

(4.3) Let $\{a, b, c\} \in \binom{E}{3}$ with $(a, b) \equiv (a, c)$ and $\{b'\} := (b \perp \overline{a, c}) \cap \overline{a, c}$ then:

(1) $(c|a, b') = 1$.
(2) $]b, c[= \{x \in \overline{b, c} | \lambda(a, x) < \lambda(a, b)\}$.

Proof. Let M, resp. m, be the midline, resp. the midpoint, of b, c. Then $a \in M$, $(M|b, c) = (m|b, c) = -1$ and if $a \notin \overline{b, c}$ then $\overline{a, m} \perp \overline{m, c}$. By [6], (20.5.1) $\{m'\} := (m \perp \overline{a, c}) \cap \overline{a, c} \subseteq]a, c[$, i.e. $(m'|a, c) = -1$ implying $(c|a, m') = 1$ by (Z2). Moreover since $(\overline{m, m'}|b, b') = 1$, we have $(m'|b', c) = (\overline{m, m'}|b', c) \cdot (\overline{m, m'}|b, b') = (\overline{m, m'}|b, c) = (m|b, c) = -1$ hence $(c|m', b') = 1$. Together: $(c|a, b') = (c|a, m') \cdot (c|m', b') = 1 \cdot 1 = 1$ showing (1) in the case $a \notin \overline{b, c}$. If $a \in \overline{b, c}$ then $a = m$, i.e. $(a|b, c) = -1$ hence by (Z2), $(c|a, b) = 1$.

(2) If $m \neq a$ then $(a, b, m), (a, c, m) \in \varDelta$ are rectangular triangles and by (4.2.1) and (4.2.4) we have for $x \in \overline{b, c}$:

$$x \in]b, c[\iff \lambda(a, x) < \lambda(a, b) = \lambda(a, c).$$

If $a = m$ then a is the midpoint of b and c and the statement is clear. □

(4.4) Let $a, b \in E$ with $a \neq b$, $S := \{x \in E | (a, x) \equiv (a, b)\}$, $T := (b \perp \overline{a, b})$ and $x, y, z \in S \backslash \{b\}$ then:

(1) $\forall t \in T \backslash \{b\} : \lambda(a, b) < \lambda(a, t)$ and $T \cap S = \{b\}$.
(2) $\forall t \in T : |\overline{a, t} \cap S| = 2$.
(3) $(T|x, y) = 1$.
(4) If $(\overline{y, b}|x, z) = -1$ then $\{u\} :=]y, b[\cap]x, z[\neq \emptyset$ and $(\overline{x, b}|y, z) = (\overline{z, b}|x, y) = 1$.
(5) If $|\{x, y, z\}| = 3$ then $(\overline{y, b}|x, z) = (\overline{x, z}|y, b)$.

Proof. Let $t \in T \backslash \{b\}$. By (4.2.1) $\lambda(a, b) < \lambda(a, t)$ hence $t \notin S$ and so $T \cap S = \{b\}$. By (W1) there are exactly two points $p, q \in \overline{a, t}$ with $(a, p) \equiv (a, q) \equiv (a, b)$ thus $\overline{a, t} \cap S = \{p, q\}$. Assume there are $x, y \in S \backslash \{b\}$ with $(T|x, y) = -1$ then $\{w\} := T \cap]x, y[\neq \emptyset$ and so by (4.3.2), $\lambda(a, w) < \lambda(a, b)$ contradicting $\lambda(a, b) < \lambda(a, w)$ (proved in (1)).

(4) Since $(\overline{y, b}|x, z) = -1$ the point $\{u\} := \overline{y, b} \cap]x, z[$ exists and by (4.3.2), $\lambda(u, a) < \lambda(a, x) = \lambda(a, b)$ hence again by (4.3.2) $u \in]y, b[$. Thus by (Z2) $1 = (b|y, u) = (\overline{b, x}|y, u) = (\overline{b, z}|y, u) = (x|u, z) = (\overline{b, x}|u, z) = (z|u, x) = (\overline{b, z}|u, x)$ and so $(\overline{x, b}|y, z) = (\overline{x, b}|y, u) \cdot (\overline{x, b}|u, z) = 1 \cdot 1 = 1$. In the same way $(\overline{z, b}|x, y) = 1$.

(5) If $(\overline{y, b}|x, z) = -1$ then $\{u\} := \overline{y, b} \cap]x, z[\subseteq]y, b[$ (cf. proof of (4)) and so $(\overline{x, z}|y, b) = (u|y, b) = -1$. If $(\overline{x, z}|y, b) = -1$ then in the same way $\{v\} := \overline{x, z} \cap]y, b[\subseteq]x, z[$ and so $(\overline{y, b}|x, z) = (v|x, z) = -1$. □

(4.5) Let $Y := \overline{o, e_2}$ and for $p \in E \backslash (W \cup Y)$ let $A := (p \perp W)$, $\{a\} := A \cap W$, $B := (p \perp Y)$, $\{b\} := B \cap Y$, $A' := (a \perp B)$, $\{a'\} := A' \cap B$, $B' := (b \perp A)$, $\{b'\} := B' \cap A$. If $(E, \mathscr{L}, \equiv, \alpha)$ is an *ordinary absolute plane* (i.e. there are no rectangles and so $A \neq A'$ and $B \neq B'$) then: $a' \in]b, p[\iff b' \in]a, p[$.

Proof. Since $W, B' \perp A$ and $W, B \perp Y$ we have $(W|b', b) = (W|b, p) = 1$ hence $(a|b', p) = (W|b', p) = (W|b', b) \cdot (W|b, p) = 1 \cdot 1 = 1$ and in the same way $(b|a', p) = 1$. Assume $b' \in \,]a, p[$ and $a' \notin \,]b, p[$. This implies $1 = (a'|b, p) = (a|b', p) = (p|b', a)$ and $(b'|a, p) = -1$. Now: $(b|a', p) = (a'|b, p) = 1$ implies firstly $(p|a', b) = -1$ (by (Z2)), i.e. $p \in]a', b[$ and $(A|a', b) = -1$, and secondly $(B'|p, a') = (b|p, a') = 1$ with $(B'|p, a) = (b'|p, a) = -1$; hence $(B'|a, a') = (B'|a, p) \cdot (B'|p, a') = (-1) \cdot 1 = -1 \Rightarrow \{b''\} := B' \cap \,]a, a'[\neq \emptyset$.

Since $(b, b'', a') \in \varDelta$ with $\overline{b, a'} = B \perp A' = \overline{b'', a'}$, $p \in \,]a', b[$ and $\{b'\} = (b \perp A) \cap A = (p \perp \overline{b, b'}) \cap \overline{b, b'}$ we have by (4.2.3) $\lambda(p, b') < \lambda(b'', a')$.

Since $(a, p, a') \in \varDelta$ with $\overline{a, a'} \perp \overline{p, a'}$ it follows by (4.2.1): $\lambda(a, a') < \lambda(a, p)$.

From $b'' \in \,]a, a'[$, resp. $b' \in \,]a, p[$, we obtain: $\lambda(a, a') = \lambda(a, b'') + \lambda(b'', a')$, resp. $\lambda(a, p) = \lambda(a, b') + \lambda(b', p)$.

Finally, since $(a, b'', b') \in \varDelta$ with $\overline{a, b'} \perp \overline{b', b''}$ we have by (4.2.1) $\lambda(a, b') < \lambda(a, b'')$. Together we obtain the contradiction: $\lambda(a, p) = \lambda(a, b') + \lambda(b', p) < \lambda(a, b'') + \lambda(b'', a') = \lambda(a, a') < \lambda(a, p)$. □

(4.6) The distance function λ satisfies the *triangular inequality*: $\forall \{a, b, c\} \in \binom{E}{3}$:

(1) $\lambda(a, b) \leqslant \lambda(a, c) + \lambda(c, b)$.
(2) $\lambda(a, b) = \lambda(a, c) + \lambda(c, b) \iff c \in \,]a, b[$.

Proof. Let $\{c'\} := (c \perp \overline{a, b}) \cap \overline{a, b}$.

Case 1: $c' = c$. By (4.1.3) if $c' = c \in \,]a, b[$ then $\lambda(a, b) = \lambda(a, c) + \lambda(c, b)$. If $a \in \,]c, b[$ or $b \in \,]c, a[$ then by (4.1.3)
$\lambda(c, b) = \lambda(a, c) + \lambda(a, b) > \lambda(a, b)$ or
$\lambda(a, c) = \lambda(b, c) + \lambda(a, b) > \lambda(a, b)$ hence in any of these cases
$\lambda(a, b) < \lambda(a, c) + \lambda(c, b)$.

Case 2: $c' \neq c$. Then by (4.2.1) $\lambda(a, c') < \lambda(a, c)$ and $\lambda(b, c') < \lambda(b, c)$ and so by case 1: $\lambda(a, b) \leqslant \lambda(a, c') + \lambda(b, c') < \lambda(a, c) + \lambda(b, c)$. □

From [6], (20.5) follows:

(4.7) Let $L \in \mathscr{L}$, $p \in E$, $p' := (p \perp L) \cap L$ and $x \in L$ then:
$\lambda(p, p') \leqslant \lambda(p, x)$ and $\lambda(p, p') = \lambda(p, x) \iff x = p'$.

(4.8) Let $A, B, C \in \mathscr{L}$ with $A \neq B$ and $A, B \perp C$, let $\{a\} := A \cap C$, $\{b\} := B \cap C$, $x \in A \backslash \{a\}$, $X := (x \perp B)$, $\{x'\} := X \cap B$, $Y := (a \perp X)$ and $\{y\} := Y \cap X$. Then:

(1) $\lambda(a, b) = \lambda(x, x') \iff y = x \iff Y = A$,
(2) $\lambda(a, b) < \lambda(x, x') \iff (Y|x, x') = -1$,
(3) $\lambda(a, b) > \lambda(x, x') \iff (Y|x, x') = 1$.

Proof. (1) From $A \neq B$ and $A, B \perp C$ follows $A \cap B = \emptyset$ and so $a \neq b$, $x \neq x'$ and $(B|a, x) = 1$. This and $Y, B \perp X$ implies $Y \cap B = \emptyset$ hence $(B|y, a) = 1$. Together we obtain:

(i) $x' \neq x, y, b$ and $(x'|x, y) = (B|x, y) = (B|x, a) \cdot (B|a, y) = 1 \cdot 1 = 1$.

Let m be the midpoint and M the midline of b and x' hence $M = (m \perp B)$ and $(M|b, x') = (m|b, x') = -1$. From $C, M, X \perp B$ follows $(M|a, b) = (M|x, x') = 1$ and so:

(ii) $(M|a, x) = (M|a, b) \cdot (M|b, x') \cdot (M|x', x) = 1 \cdot (-1) \cdot 1 = -1$ i.e. the point $\{r\} := M \cap \,]a, x[$ exists with $(r|a, x) = -1$.

Let $R := (r \perp X)$ and $\{r'\} := R \cap X$. Then $r \in A \cap R$, $A \cap B = \emptyset$, $R, B \perp X$ and (i) imply:

(iii) $r' \neq x'$ and $(B|r', x) = (B|r', r) \cdot (B|r, x) = 1 \cdot 1 = 1$, i.e. $r' \in \overrightarrow{x', x}$.

Since $\widetilde{M}$ is a motion with $\widetilde{M}(b) = x'$ and $\widetilde{M}(C) = X$ we have $\widetilde{M}(\{a\}) = \widetilde{M}((r \perp C) \cap C) = (r \perp X) \cap X = R \cap X = \{r'\}$, i.e. $\widetilde{M}(a) = r'$ and $\lambda(a, b) = \lambda(r', x')$. Therefore by (iii): $\lambda(x, x') = \lambda(a, b) \iff x = r' \iff R = A \iff Y = A$, i.e. (1) is proved.

(2) Now let $x \neq r'$, i.e. $R \neq A$ and so $(R|a, x) = (r|a, x) = -1$ by (ii). By $R, Y \perp X$ we obtain $(R|a, y) = 1$ and so:

(iv) $(r'|x, y) = (R|x, y) = (R|a, x) \cdot (R|a, y) = (-1) \cdot 1 = -1$ hence $\widetilde{M}(a) = r' \in]x, y[$.

The properties $(B|x, y) = 1$ (cf. (i)) and $r' \in]x, y[$ imply:

(v) $(B|x, r') = (B|y, r') = 1$ and $(x'|x, r') = (x'|y, r') = 1$ hence $(x|r', x') = (x|y, r') \cdot (x|y, x') = (x|y, x')$.

If $(y|x, x') = (Y|x, x') = -1$ then $(x|y, x') = 1$ and so $(x|r', x') = 1$. Now $(x'|x, r') = (x|r', x') = 1$ implies $(r'|x, x') = -1$, i.e. $r' \in]x, x'[$ and so $\lambda(a, b) = \lambda(r', x') < \lambda(x, x')$.

(3) If $(y|x, x') = 1$ then by (i), $(x|x', y) = -1$ hence by (v), $(x|r', x') = -1$, i.e. $x \in]r', x'[$ and so $\lambda(a, b) = \lambda(r', x') > \lambda(x, x')$. $\square$

5. Induced separation and cyclic order on circles

Firstly, we recall the notions: *separation*, *separated group*, *cyclic order* and *cyclic ordered group* (cf. [4,5]).

Let S be a nonempty set and let $S^{(4)} := \{(a, b, c, d) \in S^4 | a, b \neq c, d\}$. A function

$$\tau : S^{(4)} \to \{1, -1\};\ (a, b, c, d) \mapsto [a, b|c, d]$$

is called *separation* on S, if the following hold:

(T1) $\forall a, b, c, d, e \in S$ with $a, b \neq c, d, e$:

$$[a, b|c, d] \cdot [a, b|d, e] = [a, b|c, e].$$

(T2) $\forall \{a, b, c, d\} \in \binom{S}{4}$: exactly one of the values $[a, b|c, d]$, $[a, c|d, b]$, $[a, d|b, c]$ is equal -1.

(T3) $\forall (a, b, c, d) \in S^{(4)} : [a, b|c, d] = [c, d|a, b]$.

If S is also provided with a group structure "$\cdot$" then $(S, \cdot, \tau)$ is a *separated group*, if moreover the following are satisfied: $\forall (a, b, c, d) \in S^{(4)},\ \forall u \in S$:

(L) $[ua, ub \mid uc, ud] = [a, b|c, d]$.
(R) $[au, bu \mid cu, du] = [a, b|c, d]$.
(I) $[a^{-1}, b^{-1}|c^{-1}, d^{-1}] = [a, b|c, d]$.

A function $\zeta : \left(\binom{S}{3}\right) \to \{1, -1\}$; $(x, y, z) \mapsto \zeta(x, y, z)$ is called *cyclic order*, if:

(C1) $\forall \{a, b, c\} \in \binom{S}{3} : \zeta(a, b, c) = \zeta(b, c, a) = -\zeta(b, a, c)$.
(C2) $\forall \{a, b, c, d\} \in \binom{S}{4} : \zeta(a, b, c) = \zeta(a, c, d) \Rightarrow \zeta(a, b, d) = \zeta(a, b, c)$.

Furthermore $(S, \cdot, \zeta)$ is a *cyclic ordered group* if $(S, \cdot)$ is a group, ζ is a cyclic order and moreover

(CL) $\forall \{a, b, c\} \in \binom{S}{3}, \forall d \in S : \zeta(da, db, dc) = \zeta(a, b, c)$.

Remark. By [3, p. 523] a cyclic ordered group has the property:

(ZG) $\forall \{a, b, c\} \in \binom{S}{3} : \zeta(a, b, c) = \zeta(c^{-1}, b^{-1}, a^{-1})$.

We have the results (cf. [1,2, 5, p. 294] in particular (1.2)):

(5.1) Let (S, ζ) be a cyclic ordered set and let

$$\zeta' : \left(\binom{S}{3}\right) \to \{1, -1\};\ (x, y, z) \mapsto \zeta'(x, y, z) := \zeta(z, y, x),$$

then:

(1) (S, ζ') is also a cyclic ordered set, the *opposite* cyclic order.

(2) There is exactly one separation τ_ζ on S such that if $\{a,b,c,d\} \in \binom{S}{4}$ then $[a,b|c,d] = \zeta(a,c,d)\cdot\zeta(b,c,d)$ and we have $\tau_\zeta = \tau_{\zeta'}$.

(5.2) Let (S,τ) be a set with a separation function, then there is exactly one pair of opposite cyclic orders ζ and ζ' such that $\tau = \tau_\zeta = \tau_{\zeta'}$.

Now we show that the order structure α of our absolute plane induces a cyclic order ζ on the commutative group $(E_1,\cdot)$ such that $(E_1,\cdot,\zeta)$ becomes a cyclic ordered group.

(5.3) Let

$$\zeta : \left(\binom{E_1}{3}\right) \to \{1,-1\};\ \ (x,y,z) \mapsto \zeta(x,y,z) := (\overline{e_1, x^{-1}z} \mid e_o, x^{-1}y).$$

Then $(E_1,\cdot,\zeta)$ is a cyclic ordered group.

Proof. Let $T_1 := (e_1 \perp W)$, $u := x^{-1}\cdot y$ and $v := x^{-1}\cdot z$ then $u, v \in E_1\backslash\{e_1\}$, $u \neq v$ and by (4.4.3), $(T_1|u,v) = 1$ hence by (1.2):

$(*)$ $\zeta(x,y,z) = (\overline{e_1,v}|e_o,u) = -(\overline{e_1,u}|e_o,v) = -\zeta(x,z,y)$.

If U is the midline of e_1 and u then $\widetilde{U} = \widehat{e_1,u}$ and $u^{\cdot} = \widetilde{U}\circ\widetilde{W}$ is order preserving (cf. (2.3.3)). Therefore we have:

$$\begin{aligned}\zeta(y,z,x) &= (\overline{e_1,u^{-1}}|e_o,u^{-1}\cdot v) = (\overline{u,e_1}|u^{\cdot}(e_o),v)\\ &= (\overline{e_1,u}|e_o,v)\cdot(\overline{e_1,u}|e_o,u^{\cdot}(e_o)) = -\zeta(x,y,z)\cdot(\overline{e_1,u}|e_o,u^{\cdot}(e_o)).\end{aligned}$$

Since $\widetilde{U}(e_1) = u$ hence $\widetilde{U}(\overline{e_1,u}) = \overline{e_1,u}$ and so by (2.2.2), $(\overline{e_1,u}|e_o,\widetilde{U}(e_o)) = 1$ we obtain

$$\begin{aligned}(\overline{e_1,u}|e_o,\widetilde{U}\circ\widetilde{W}(e_o)) &= (\overline{e_1,u}|\widetilde{U}(e_o),\widetilde{W}(e_o))\\ &= (\overline{e_1,u}|e_o,\widetilde{U}(e_o))\cdot(\overline{e_1,u}|e_o,\widetilde{W}(e_o)) = (\overline{e_1,u}|e_o,\widetilde{W}(e_o)).\end{aligned}$$

By (O4) and (2.3.1),

$(\overline{e_1,u}|e_o,\widetilde{W}(e_o)) = (W|e_o,\widetilde{W}(e_o)) = -1$.

Thus together $\zeta(y,z,x) = \zeta(x,y,z)$ and so with $(*)$ the condition (C1) is verified.

By the definition of ζ the condition (CL) is valid. Therefore in order to prove (C2) we may assume $a = e_1$. Then $\zeta(e_1,b,c) = (\overline{e_1,c}|e_o,b) = \zeta(e_1,c,d) = (\overline{e_1,d}|e_o,c) = -(\overline{e_1,c}|e_o,d)$ implies $(\overline{e_1,c}|b,d) = (\overline{e_1,c}|e_o,b)\cdot(\overline{e_1,c}|e_o,d) = -1$ and so by (4.4.4), $(\overline{e_1,b}|c,d) = (\overline{e_1,d}|b,c) = 1$. Now (by using (C1)) $\zeta(e_1,b,d)\cdot\zeta(e_1,b,c) = \zeta(e_1,d,b)\cdot\zeta(e_1,c,b) = (\overline{e_1,b}|e_o,d)\cdot(\overline{e_1,b}|e_o,c) = (\overline{e_1,b}|c,d) = 1$ implies $\zeta(e_1,b,d) = \zeta(e_1,b,c)$. □

By [5], (1.10) since $-e_1$ is the only involution of $(E_1,\cdot)$ we have:

(5.4) Let $b \in E_1^+ := \{x \in E_1|\zeta(e_1,x,-e_1) = 1\} = \{x \in E_1|(W|e_o,x) = 1\}$ and $\zeta(e_1,a,b) = 1$, then $a \in E_1^+$.

(5.5) For $\{x,y,z\} \in \binom{E_1\backslash\{e_1\}}{3}$:

$$\zeta(x,y,z) = \zeta(e_1,y,x)\cdot\zeta(e_1,z,y)\cdot\zeta(e_1,x,z).$$

Proof. By (C1) and (CL) we have

$$\begin{aligned}\zeta(x,y,z)\cdot\zeta(e_1,x,z) &= \zeta(e_1,x^{-1}y,x^{-1}z)\cdot\zeta(e_1,x^{-1}z,x^{-1})\\ &= -\zeta(e_1,x^{-1}y,x^{-1}z)\cdot\zeta(e_1,x^{-1},x^{-1}z)\\ &= -(\overline{e_1,x^{-1}z}|e_o,x^{-1}y)\cdot(\overline{e_1,x^{-1}z}|e_o,x^{-1})\\ &= -(\overline{e_1,x^{-1}z}|x^{-1}y,x^{-1}) = -(\overline{x,z}|y,e_1)\end{aligned}$$

and

$$\begin{aligned}\zeta(e_1,y,x)\cdot\zeta(e_1,z,y) &= -\zeta(e_1,x,y)\cdot\zeta(e_1,z,y)\\ &= -(\overline{e_1,y}|e_o,x)\cdot(\overline{e_1,y}|e_o,z) = -(\overline{e_1,y}|x,z).\end{aligned}$$

With (4.4.5) we have the assertion. □

(5.6) For $x, y \in E_1$ with $x \neq y$ let: $x < y$: if $x, y \neq e_1$ and $\zeta(e_1, x, y) = (\overline{e_1, y}|e_o, x) = 1$ or if $x = e_1$; then: $(E_1, <)$ is a linear ordered set and for $\{a, b, c\} \in \binom{E_1}{3}$ we have $\zeta(a, b, c) = 1 \iff a < b < c \ \vee \ b < c < a \ \vee \ c < a < b$.

Proof. Let $x, y \in E_1\backslash\{e_1\}$ with $x \neq y$ then $\zeta(e_1, x, y) = -\zeta(e_1, y, x)$ hence either $x < y$ or $y < x$. Now let $x, y, z \in E_1\backslash\{e_1\}$ with $x < y$ and $y < z$, i.e. $\zeta(e_1, x, y) = \zeta(e_1, y, z) = 1$. Then by (C2), $\zeta(e_1, x, z) = \zeta(e_1, x, y) = 1$, hence $x < z$. The second part is a direct consequence of the first part and statement (5.5). $\square$

(5.7) For $a, b \in E_1\backslash\{e_1\}$ with $a \neq b$ the following conditions are equivalent:

(1) $\zeta(e_1, a, b) = 1$.
(2) At least one of the sets $\{a, b^{-1}\}$, $\{a, b, a^{-1}b\}$, $\{a^{-1}, b^{-1}, a^{-1}b\}$ is contained in $\overline{E_1^+} := E_1^+ \cup \{-e_1\}$.

Proof. Let $x \in E_1\backslash\{e_1, -e_1\}$ then by (4.4.3) $(\overline{e_1, e_o}|x, -e_1) = 1$ and so by [6], (13.12), $(\overline{e_1, x}|-e_1, e_o) = -(\overline{e_1, -e_1}|x, e_o)$ hence:

$(*)$ $x \in E_1^+ \iff \zeta(e_1, x, -e_1) = -(\overline{e_1, x}| - e_1, e_o) = 1$ and
$x \in E_1^- := \{x \in E_1 \mid (W|e_o, x) = -1\} \iff \zeta(e_1, x, -e_1) = -(\overline{e_1, x}| - e_1, e_o) = -1$.
By (ZG),(C1) and (CL) we have:
$(**)$ $\zeta(e_1, a, b) = \zeta(b^{-1}, a^{-1}, e_1) = \zeta(e_1, b^{-1}, a^{-1}) = \zeta(e_1, a^{-1}b, b)$.
Case 1: $b \in \overline{E_1^+}$.
If $\zeta(e_1, a, b) = 1$ then by $(**)$ and (5.4), $\{a, b, a^{-1}b\} \subseteq \overline{E_1^+}$.
If $\{a, b, a^{-1}b\} \subseteq \overline{E_1^+}$ and $\zeta(e_1, a, b) \neq 1$ then $\zeta(e_1, b, a) = 1$ and so $b^{-1}a = (a^{-1}b)^{-1} \in \overline{E_1^+}$ hence $a^{-1}b \in \overline{E_1^+} \cap (\overline{E_1^+})^{-1} = \{-e_1\}$, i.e. $b = (-e_1) \cdot a \in \overline{E_1^+} \cap (\overline{E_1^+})^{-1} = \{-e_1\}$. But $(-e_1) \cdot a = -e_1$ implies the contradiction $a = e_1$. Consequently if $b \in \overline{E_1^+}$:
$\zeta(e_1, a, b) = 1 \iff \{a, b, a^{-1}b\} \subseteq \overline{E_1^+}$.
Case 2: $a \in \overline{E_1^-}$ hence $a^{-1} \in \overline{E_1^+}$. Then by $(**)$, (ZG) and Case 1:
$1 = \zeta(e_1, a, b) = \zeta(e_1, b^{-1}, a^{-1}) \iff \{b^{-1}, a^{-1}, ba^{-1}\} \subseteq \overline{E_1^+}$.
Case 3: $a \in E_1^+ \wedge b \in E_1^-$.
Since $W := \overline{e_1, -e_1}$, $(W|a, b) = (W|a, e_o) \cdot (W|b, e_o) = 1 \cdot (-1) = -1$ and by (4.4.4), $\{p\} :=]a, b[\cap]e_1, -e_1[\neq \emptyset$, hence $(e_1|-e_1, p) = (b|p, a) = 1$ and so $(\overline{e_1, b}|-e_1, a) = (\overline{e_1, b}|-e_1, p) \cdot (\overline{e_1, b}|p, a) = (e_1|-e_1, p) \cdot (b|p, a) = 1 \cdot 1 = 1$. Since $b \in E_1^-$ by $(*)$, $(\overline{e_1, b}|-e_1, e_o) = 1$. Finally, $\zeta(e_1, a, b) = (\overline{e_1, b}|a, e_o) = (\overline{e_1, b}|-e_1, a) \cdot (\overline{e_1, b}|-e_1, e_o) = 1 \cdot 1 = 1$.
This shows that (1) and (2) are equivalent. $\square$

From (5.7) and (CL) we obtain:

(5.8) Let $\{a, b, c\} \in \binom{E_1}{3}$ then the following conditions are equivalent:

(1) $\zeta(a, b, c) = 1$,
(2) at least one of the sets $\{a^{-1}b, c^{-1}a\}$, $\{a^{-1}b, a^{-1}c, b^{-1}c\}$, $\{ab^{-1}, ac^{-1}, b^{-1}c\}$ is contained in E_1^+.

6. Angles and measures of angles

By an *angle* $\measuredangle(a, b, c)$ we understand an ordered pair of halflines $(\overrightarrow{b, a}, \overrightarrow{b, c})$ having the initial point in common. If $\measuredangle(a, b, c)$ and $\measuredangle(d, e, f)$ are two angles we can find on the halflines $\overrightarrow{b, a}, \overrightarrow{b, c}, \overrightarrow{e, d}, \overrightarrow{e, f}$ points a', c', d', f' such that $(b, a') \equiv (b, c') \equiv (e, d') \equiv (e, f') \equiv (o, e_1)$. Then $\measuredangle(a, b, c)$ and $\measuredangle(d, e, f)$ are *congruent* if $(a', c') \equiv (d', f')$ (cf. [6, p. 98]) or equivalently if there is a motion $\sigma \in \mathscr{M}$ with $\sigma(\overrightarrow{b, a}) = \overrightarrow{\sigma(b), \sigma(a)} = \overrightarrow{e, d}$ and $\sigma(\overrightarrow{b, c}) = \overrightarrow{e, f}$. If even $\sigma \in \mathscr{M}^+$ then we call $\measuredangle(a, b, c)$ and $\measuredangle(d, e, f)$ *conjugate* and express that by $\measuredangle(a, b, c) =^\circ \measuredangle(e, d, f)$.

Like in Section 4 if $a, b \in E$ with $a \neq b$ let $\sigma_{a,b} \in \mathscr{M}^+$ be the proper motion such that $\sigma_{a,b}(a) = o$ and $\sigma_{a,b}(b) \in W_+ := \overrightarrow{o, e_1}$. Let $\mathscr{A}$ be the set of all angles. Then the function

$$\mu : \mathscr{A} \to E_1; \quad \measuredangle(a, b, c) \mapsto \sigma_{b,a}(\overrightarrow{b, c}) \cap E_1$$

is called the *oriented measure*. Two angles $\alpha := \sphericalangle(a,b,c)$ and $\beta := \sphericalangle(d,e,f)$ are *conjugate* if and only if $\mu(\alpha)=\mu(\beta)$ and *congruent* if and only if $\mu(\beta) \in \{\mu(\alpha), (\mu(\alpha))^{-1}\}$. We have $(\mu(\alpha))^{-1} = \mu(\sphericalangle(c,b,a))$ and call α *positive*, resp. *elongated* or resp. *negative*, if $\mu(\alpha) \in E_1^+$, resp. $\mu(\alpha) = -e_1$ or resp. $\mu(\alpha) \in E_1^-$. If α is positive then α is *acute*, resp. *right-angled*, resp. *obtuse* if $\zeta(e_1, \mu(\alpha), e_2) = 1$, resp. $\mu(\alpha) = e_2$, resp. $\zeta(e_2, \mu(\alpha), -e_1) = 1$. The angles α and β can be added if $\overrightarrow{b,c} = \overrightarrow{e,d}$ and then $\alpha + \beta := \sphericalangle(a,b,f)$ and $\mu(\alpha+\beta) = \mu(\alpha)\cdot\mu(\beta)$. If α and β are positive then $\zeta(e_1, \mu(\alpha), \mu(\alpha+\beta)) = \zeta(e_1, \mu(\beta), \mu(\alpha+\beta)) = 1$.

References

[1] D. Gröger, Archimedisierung absoluter Ebenen, J. Geom. 25 (1985) 49–76.

[2] D. Gröger, Archimedisierung elliptischer Ebenen, Mitt. Math. Ges. Hamburg 11 (Heft 4) (1987) 441–457.

[3] H. Karzel, Zyklisch geordnete Gruppen, Mitt. Math. Ges. Hamburg 10 (Heft 7) (1979) 523–529.

[4] H. Karzel, Recent developments on absolute geometries and algebraization by K-loops, Discrete Math. 208, 209 (1999) 387–409.

[5] H. Karzel, C.J. Maxson, Archimedeisation of some ordered geometric structures which are related to kinematic spaces, Res. Math. 19 (1991) 290–318.

[6] H. Karzel, K. Sörensen, D. Windelberg, Einführung in die Geometrie, UTB 184, Vandenhoeck, Göttingen, 1973.

[7] H. Kiechle, Theory of K-Loops, Lecture Notes in Mathematics, vol. 1778, Springer, Berlin, 2002.

[8] K. Sörensen, Ebenen mit Kongruenz, J. Geom. 22 (1984) 15–30.

[9] E. Sperner, Die Ordnungsfunktionen einer Geometrie, Math. Ann. 121 (1949) 107–130 Reprinted in: H. Karzel, K. Sörensen, Wandel von Begriffsbildungen in der Mathematik, Darmstadt, 1984, pp. 138–161.